On b-continuity of Kneser Graphs of Type $KG(2k+1,k)$

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Abstract

In this paper, we will introduce an special kind of graph homomorphisms namely semi-locally-surjective graph homomorphisms and show some relations between semi-locally-surjective graph homomorphisms and colorful colorings of graphs and then we prove that for each natural number k, the Kneser graph $KG(2k+1,k)$ is b-continuous. Finally, we introduce some special conditions for graphs to be b-continuous.

Keywords: graph colorings, colorful colorings, Kneser graphs, semi-locally-surjective graph homomorphisms.
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1 Introduction

All graphs considered in this paper are finite and simple (undirected, loopless and without multiple edges). Let $G=(V,E)$ be a graph and $k \in \mathbb{N}$ and let $[k] := \{i \mid i \in \mathbb{N},\ 1 \leq i \leq k\}$. A k-coloring (proper k-coloring) of G is a function $f : V \to [k]$ such that for each $1 \leq i \leq k$, $f^{-1}(i)$ is an independent set. We say that G is k-colorable whenever G has a k-coloring f, in this case, we denote $f^{-1}(i)$ by V_i and call each $1 \leq i \leq k$, a color (of f) and each V_i, a color class (of f). The minimum integer k for which G has a k-coloring, is called the chromatic number of G and is denoted by $\chi(G)$.

Let G be a graph and f be a k-coloring of G and v be a vertex of G. The vertex v is called b-dominating (or colorful or color-dominating) (with respect to f) if each color $1 \leq i \leq k$ appears on the closed neighborhood of v ($f(N[v]) = [k]$). The coloring f is said to be a colorful k-coloring of G if each color class V_i ($1 \leq i \leq k$) contains a b-dominating vertex x_i. Obviously, every $\chi(G)$-coloring of G is a colorful $\chi(G)$-coloring of G. We denote $\mathrm{B}(G)$ the set of all positive integers k for which G has a colorful k-coloring. The maximum of $\mathrm{B}(G)$, is called the b-chromatic number of G and is denoted by $b(G)$ (or $\phi(G)$ or $\chi_b(G)$). The graph G is said to be b-continuous if each integer k between $\chi(G)$ and $b(G)$ is an element of $\mathrm{B}(G)$.

There are graphs that are not b-continuous, for example, the 3-dimensional cube Q_3 is not b-continuous, because $2 \in \mathrm{B}(G)$ and $4 \in \mathrm{B}(G)$ but $3 \notin \mathrm{B}(G)$ ([5]). We have to note that the problem of deciding whether graph G is b-continuous is NP-complete ([1]). The colorful coloring of graphs was introduced in 1999 in [5] with the terminology b-coloring.

Let $m, n \in \mathbb{N}$ and $m \leq n$. $KG(n, m)$ is the graph whose vertex set is the set of all subsets of size m of $[n]$ in which two vertices X and Y are adjacent iff $X \bigcap Y = \emptyset$. Note that $KG(5, 2)$ is the famous Petersen graph. It was conjectured by Kneser in 1955 ([7]), and proved by Lovász in 1978 ([8]), that if $n \geq 2m$, then $\chi(KG(n, m)) = n - 2m + 2$. Lovász's proof was the beginning of using algebraic topology in combinatorics. Colorful colorings of Kneser graphs have been investigated in [4] and [6]. Javadi and Omoomi in [6] showed that for $n \geq 17$, $KG(n, 2)$ is b-continuous. Only a few classes of graphs are known to be b-continuous (see [1], [3] and [6]). We want to prove that for each natural number k, $KG(2k + 1, k)$ is b-continuous. In this regard, first we introduce an special kind of graph homomorphisms which is related to colorful colorings of graphs.

Definition 1. Let G and H be graphs. A function $f : V(G) \to V(H)$ is called a semi-locally-surjective graph homomorphism from G to H if f is a surjective graph homomorphism from G to H and satisfies the following condition :

$$\forall u \in V(H) : \ \exists a \in f^{-1}(u) \ s.t \ \forall v \in N_H(u) : \ \exists b \in f^{-1}(v) \ s.t \ \{a, b\} \in E(G).$$

♠

We know that a graph G is k-colorable iff there exists a graph homomorphism from G to the complete graph K_k and the chromatic number of G is the least natural number k for which there exists a graph homomorphism from G to K_k. Indeed, we can think of graph homomorphisms from graphs to complete graphs instead of graph colorings. The following obvious theorem shows such a similar relation between colorful colorings of graphs and semi-locally-surjective graph homomorphisms. Indeed, we can think of semi-locally-surjective graph homomorphisms from graphs to complete graphs instead of colorful colorings of graphs.

Theorem 1. *Let G be a graph and $k \in \mathbb{N}$. Then $k \in \mathrm{B}(G)$ iff there exists a semi-locally-surjective graph homomorphism from G to K_k. Also, the chromatic number of G ($\chi(G)$) and the b-chromatic number of G ($b(G)$) are respectively the least and the greatest natural numbers k for which there exists a semi-locally-surjective graph homomorphism from G to K_k .*

We know that the composition of two graph homomorphisms is again a graph homomorphism. A similar theorem holds for composition of semi-locally-surjective graph homomorphisms.

Theorem 2. *Let G_1, G_2 and G_3 be graphs. If g is a semi-locally-surjective graph homomorphism from G_2 to G_1 and f is a semi-locally-surjective graph homomorphism from G_3 to G_2, then gof is a semi-locally-surjective graph homomorphism from G_3 to G_1.*

The following theorem shows another relation between semi-locally-surjective graph homomorphisms and colorful colorings of graphs.

Theorem 3. *Let G_1 and G_2 be graphs. If there exists a semi-locally-surjective graph homomorphism from G_1 to G_2, then $\mathrm{B}(G_2) \subseteq \mathrm{B}(G_1)$.*

Proof. Let f be a semi-locally-surjective graph homomorphism from G_1 to G_2, $k \in \mathrm{B}(G_2)$, and $V_1, \ldots, V_k$ be color classes of a colorful k-coloring of G_2 and $x_1, \ldots, x_k$ be some b-dominating vertices of G_2 with respect to this k-coloring and $x_i \in V_i$ $(1 \leq i \leq k)$. Obviously, $f^{-1}(V_1), \ldots, f^{-1}(V_k)$ are nonempty color classes of a k-coloring of G_1 and $f^{-1}(x_1), \ldots, f^{-1}(x_k)$ are some b-dominating vertices of G_1 with respect to this k-coloring and $f^{-1}(x_i) \in f^{-1}(V_i)$ $(1 \leq i \leq k)$. Therefore, G_1 has a colorful k-coloring and $k \in \mathrm{B}(G_1)$. Hence, $\mathrm{B}(G_2) \subseteq \mathrm{B}(G_1)$. ∎

Now we prove that for each natural number k, $KG(2k+1, k)$ is b-continuous.

Theorem 4. *For each $k \in \mathbb{N}$, $KG(2k+1, k)$ is b-continuous.*

Proof. For each $k \in \mathbb{N}$, $\chi(KG(2k+1,k)) = 3$. Note that $\mathrm{B}(KG(3,1)) = \mathrm{B}(K_3) = \{3\}$ and therefore, for $k = 1$ the assertion follows. Blidia, et al. in [2] proved that the b-chromatic number of the Petersen graph is 3 and therefore, $\mathrm{B}(KG(5,2)) = \mathrm{B}(\textit{Petersen graph}) = \{3\}$. Hence, $KG(2k+1,k)$ is b-continuous for $k = 2$. For $k \geq 3$, the function $f : V(KG(2k+3, k+1)) \to V(KG(2k+1,k))$ which assigns to each $A \subseteq [2k+3]$ with $|A \bigcap \{2k+2, 2k+3\}| \leq 1$, $f(A) = A \setminus \{\max A\}$ and to each $A \subseteq [2k+3]$ with $\{2k+2, 2k+3\} \subseteq A$, $f(A) = (A \setminus \{2k+2, 2k+3\}) \bigcup \{\max([2k+1] \setminus A)\}$, is a surjective graph homomorphism from $KG(2k+3, k+1)$ to $KG(2k+1, k)$. Now for each $X \in V(KG(2k+1,k))$, $(X \bigcup \{2k+2\}) \in f^{-1}(X)$ and for each $Y \in N_{KG(2k+1,k)}(X)$, $(Y \bigcup \{2k+3\}) \in f^{-1}(Y)$ and $\{X \bigcup \{2k+2\}, Y \bigcup \{2k+3\}\} \in E(KG(2k+3, k+1))$. Hence, f is a semi-locally-surjective graph homomorphism from $KG(2k+3, k+1)$ to $KG(2k+1, k)$. Consequently, Theorem 3 implies that $\mathrm{B}(KG(2k+1,k)) \subseteq \mathrm{B}(KG(2k+3, k+1))$, besides, $\mathrm{B}(KG(7,3)) \subseteq \mathrm{B}(KG(9,4)) \subseteq \ldots \subseteq \mathrm{B}(KG(2n+1,n)) \subseteq \ldots$. (I)

On the other hand, Javadi and Omoomi in [6] showed that for $k \geq 3$, $b(KG(2k+1,k)) = k+2$ and $k+2 \in \mathrm{B}(KG(2k+1,k))$. Therefore, for each $k \geq 3$, $\{i+2 |\ i \in \mathbb{N},\ 3 \leq i \leq k\} \subseteq \mathrm{B}(KG(2k+1,k))$. Also, since $\chi(KG(2k+1,k)) = 3$, $3 \in \mathrm{B}(KG(2k+1,k))$. So, constructing a colorful

4-coloring of $KG(2k+1,k)$ $(k \geq 3)$ completes the proof. (I) implies that it is enough to construct a colorful 4-coloring of $KG(7,3)$. Set

$$\begin{aligned}
V_1 &:= \{ \{1,2,3\},\{1,4,5\},\{2,5,6\},\{1,2,6\},\{1,2,7\},\{1,3,6\},\{1,6,7\}, \\
&\qquad \{1,4,6\} \}, \\
V_2 &:= \{ \{5,x,y\} \mid x,y \in \{1,2,3,4,6,7\},\ x \neq y \} \setminus \{ \{1,4,5\},\{2,5,6\}, \\
&\qquad \{4,5,7\} \}, \\
V_3 &:= \{ \{1,2,4\},\{1,3,7\},\{4,5,7\},\{1,4,7\},\{2,6,7\} \}, \\
V_4 &:= (\{ \{4,x,y\} \mid x,y \in \{1,2,3,6,7\},\ x \neq y\} \setminus \{ \{1,2,4\},\{1,4,6\}, \\
&\qquad \{1,4,7\} \}) \bigcup \{ \{2,3,6\},\{2,3,7\},\{3,6,7\} \}.
\end{aligned}$$

Now, one can check that $V_1,\ V_2,\ V_3,\ V_4$ are color classes of a colorful 4-coloring of $KG(7,3)$ that $\{1,2,3\} \in V_1,\ \{5,6,7\} \in V_2,\ \{2,6,7\} \in V_3$ and $\{1,3,4\} \in V_4$ are some b-dominating vertices with respect to this 4-coloring. ■

The semi-locally-surjective graph homomorphism f in above Theorem can be generalized as follows.

Theorem 5. *Let $n,m \in \mathbb{N}$ with $n > 2m$. Then* $\mathrm{B}(KG(n,m)) \subseteq \mathrm{B}(KG(n+2,m+1))$.

Proof. The function $f : V(KG(n+2,m+1)) \to V(KG(n,m))$ which assigns to each $A \subseteq [n+2]$ with $|A \bigcap \{n+1,n+2\}| \leq 1$, $f(A) = A \setminus \{\max A\}$ and to each $A \subseteq [n+2]$ with $\{n+1,n+2\} \subseteq A$, $f(A) = (A \setminus \{n+1,n+2\}) \bigcup \{\max([n] \setminus A)\}$, is a surjective graph homomorphism from $KG(n+2,m+1)$ to $KG(n,m)$. Now for each $X \in V(KG(n,m))$, $(X \bigcup \{n+1\}) \in f^{-1}(X)$ and for each $Y \in N_{KG(n,m)}(X)$, $(Y \bigcup \{n+2\}) \in f^{-1}(Y)$ and $\{X \bigcup \{n+1\}, Y \bigcup \{n+2\}\} \in E(KG(n+2,m+1))$. Hence, f is a semi-locally-surjective graph homomorphism from $KG(n+2,m+1)$ to $KG(n,m)$ and therefore, Theorem 3 implies that $\mathrm{B}(KG(n,m)) \subseteq \mathrm{B}(KG(n+2,m+1))$. ■

Corollary 1. *Let $a,b \in \mathbb{N} \bigcup \{0\}$ and $a > 2b$. Also, for each $i \in \mathbb{N} \setminus \{1\}$, let* $\mathrm{B}_i := \mathrm{B}(KG(2i+a,i+b))$ *and* $\mathrm{b_i} := b(KG(2i+a,i+b))$. *Then* $\mathrm{B}_2 \subseteq \mathrm{B}_3 \subseteq \mathrm{B}_4 \subseteq \ldots \subseteq \mathrm{B}_n \subseteq \mathrm{B}_{n+1} \subseteq \ldots$ *, and* $\mathrm{b_2} \leq \mathrm{b_3} \leq \mathrm{b_4} \leq \ldots \leq \mathrm{b_n} \leq \mathrm{b_{n+1}} \leq \ldots$.

Now we introduce some special conditions for graphs to be b-continuous. But first we note that in a graph G with at least one cycle, the girth of G $(g(G))$, is the minimum of all cycle lengths of G and if G has not any cycles, the girth of G is defined $g(G) = +\infty$.

Blidia, et al. proved the following theorem.

Theorem 6. *([2]) If $d \leq 6$, then for every d-regular graph G with girth $g(G) \geq 5$ which is different from the Petersen graph, $b(G) = d+1$.*

By using this theorem, we prove the following theorem.

Theorem 7. *Let $3 \leq d \leq 6$ and for each $2 \leq i \leq d$, G_i be an i-regular graph with girth $g(G_i) \geq 5$ which is different from the Petersen graph. Also, suppose that for each $3 \leq i \leq d$, there exists a semi-locally-surjective graph homomorphism f_i from G_i to G_{i-1}. Then for each $2 \leq i \leq d$, G_i is b-continuous.*

Proof. Theorem 6 implies that for each $2 \leq i \leq d$, $b(G_i) = i + 1$ and therefore, $i + 1 \in \mathrm{B}(G_i)$. Also, since for each $3 \leq i \leq d$, there exists a semi-locally-surjective graph homomorphism f_i from G_i to G_{i-1}, theorem 3 implies that $\mathrm{B}(G_{i-1}) \subseteq \mathrm{B}(G_i)$ and consequently, $\mathrm{B}(G_2) \subseteq \mathrm{B}(G_3) \subseteq ... \subseteq \mathrm{B}(G_d)$. Hence, for each $2 \leq i \leq d$, $\{j+1|2 \leq j \leq i\} \subseteq \mathrm{B}(G_i)$ and therefore, $\{3, 4, ..., i + 1\} \subseteq \mathrm{B}(G_i)$. Now, there are 2 cases:

Case 1) The case that G_i is bipartite. In this case, $\chi(G_i) = 2$ and therefore, $2 \in \mathrm{B}(G_i)$ and $\{2, 3, ..., i+1\} \subseteq \mathrm{B}(G_i)$, so $\mathrm{B}(G_i) = \{2, 3, ..., i+1\}$ and G_i is b-continuous.

Case 2) The case that G_i is not bipartite. In this case, $\chi(G_i) \geq 3$ and since $\{3, ..., i+1\} \subseteq B(G_i)$, so $\mathrm{B}(G_i) = \{3, ..., i+1\}$ and G_i is b-continuous.

Therefore, for each $2 \leq i \leq d$, G_i is b-continuous. ■

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